

# Multilinear generalized Radon transforms and point configurations

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**ABSTRACT.** We study multilinear generalized Radon transforms using a graph-theoretic paradigm that includes the widely studied linear case. These provide a general mechanism to study Falconer-type problems involving  $(k+1)$ -point configurations in geometric measure theory, with  $k \geq 2$ , including the distribution of simplices, volumes and angles determined by the points of fractal subsets  $E \subset \mathbb{R}^d$ ,  $d \geq 2$ . If  $T_k(E)$  denotes the set of noncongruent  $(k+1)$ -point configurations determined by  $E$ , we show that if the Hausdorff dimension of  $E$  is greater than  $d - \frac{d-1}{2k}$ , then the  $\binom{k+1}{2}$ -dimensional Lebesgue measure of  $T_k(E)$  is positive. This compliments previous work on the Falconer conjecture ([5] and the references there), as well as work on finite point configurations [6, 10]. We also give applications to Erdős-type problems in discrete geometry and a fractal regular value theorem, providing a multilinear framework for the results in [7].

## 1. Introduction

Linear generalized Radon transforms are operators of the form

$$(1.1) \quad \mathcal{R}f(x) = \int_{\{\Phi(x,y)=\vec{t}\}} f(y)\Psi(x,y)d\sigma_{\vec{t}}^x(y),$$

where  $\Phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^m$ ,  $d \geq 2$ , is a family of smooth defining functions,  $\Psi$  is a smooth cut-off function, and  $d\sigma_{\vec{t}}^x(y)$  on  $\{y : \Phi(x,y) = \vec{t}\}$  is induced from the Leray measure on the incidence relation  $\Sigma_{\vec{t}} := \{(x,y) : \Phi(x,y) = \vec{t}\}$ . All of these objects, and the operator  $\mathcal{R}$ , vary smoothly as  $\vec{t}$  varies over regular values of  $\Phi$ . Operators on this level of generality were introduced by Guillemin and Sternberg [16] and Phong and Stein [33], building on earlier work of Gelfand and his collaborators, and Helgason. Beyond their original role in integral geometry, generalized Radon transforms and their singular variants have since become ubiquitous in harmonic analysis, partial differential equations and related areas [34, 35, 36, 37, 15], and more recently as tools to study geometric and combinatorial problems [8, 5, 22].

Model cases of generalized Radon transforms have been present in the literature for a long time. An example of particular importance to this paper is the spherical averaging operator,

$$(1.2) \quad A_1^d f(x) = \int_{S^{d-1}} f(x-y)d\sigma(y),$$

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$d \geq 2$ , where  $d\sigma$  is the Lebesgue measure on the unit sphere  $S^{d-1}$ . It was proved in [29, 40] that

$$(1.3) \quad A_1^d : L^2(\mathbb{R}^d) \rightarrow L^2_{\frac{d-1}{2}}(\mathbb{R}^d),$$

where  $L_s^2(\mathbb{R}^d)$  is the standard Sobolev space consisting of  $L^2(\mathbb{R}^d)$  functions with  $s$  generalized derivatives in  $L^2(\mathbb{R}^d)$ , and that

$$(1.4) \quad A_1^d : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d)$$

if and only if  $(\frac{1}{p}, \frac{1}{q})$  belongs to the closed triangle with the endpoints  $(0, 0)$ ,  $(1, 1)$  and  $(\frac{d}{d+1}, \frac{1}{d+1})$ . This led to ongoing studies of  $L^p$ -improving properties of measures; see, e.g., [3, 31, 42] and the references there. Among many other applications,  $A_1^d$  is closely related to the fundamental solution of the wave equation; see [26, 38] and the references there.

The purpose of this paper is to introduce a class of multilinear generalized Radon transforms and apply them to several problems in harmonic analysis, geometric measure theory and discrete geometry. Before initiating this study, let us reinterpret the definition of the (linear) generalized Radon transforms in terms of a straight forward graph-theoretic paradigm. Given  $\Phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^m$  as in (1.1), define a (directed) graph whose vertices are points in  $\mathbb{R}^d$ , by saying that two vertices,  $x, y \in \mathbb{R}^d$ , are connected by an *edge*, or  $x \sim y$ , if  $\Phi(x, y) = \vec{t}$ . Recall that the adjacency operator for a finite graph  $G$  is defined by

$$Af(x) = \sum_{x \sim y} f(y).$$

Thus, the generalized Radon transform (1.1) may be viewed as a continuous analogue of the adjacency operator for the infinite directed graph defined by the pair  $(\Phi, \vec{t})$ .

The graph-theoretic perspective on generalized Radon transforms is implicit in the work of Falconer [8] and subsequent efforts on the Falconer distance problem. The question there is to determine how large the Hausdorff dimension of a subset  $E \subset \mathbb{R}^d$ ,  $d \geq 2$ , needs to be to ensure that the Lebesgue measure of the set of pairwise distances is positive. This means that one must show that a given distance cannot arise too often, either in a point-wise or average sense. If we view the points of the ambient set as vertices of a graph and connect two vertices by an edge if they are separated by a given fixed distance, then the problem is to obtain a suitable bound on the distribution of edges of this graph. This naturally leads one to the examination of the Sobolev bounds for the adjacency operator which, in this case, turns out to be the spherical averaging operator defined in (1.2).

The distance problem can be viewed as a geometric problem on two-point configurations in subsets of  $\mathbb{R}^d$ ; in this paper, we introduce  $k$ -multilinear variants of generalized Radon transforms which can be used to study  $(k+1)$ -point configurations,  $k \geq 2$ .

**1.1. Definition of a multilinear generalized Radon transform.** The graph-theoretic point of view leads naturally to the definition of multilinear generalized Radon transforms. Consider a (directed) hyper-graph whose vertices are points in  $\mathbb{R}^d$ . For  $1 \leq k \leq d$ , let  $\Phi : (\mathbb{R}^d)^{k+1} \rightarrow \mathbb{R}^m$  and  $\vec{t} \in \mathbb{R}^m$ . Then, we say that the ordered  $(k+1)$ -tuple of vectors  $(x^1, x^2, \dots, x^{k+1})$  are connected by a *hyper-edge* if  $\Phi(x^1, \dots, x^{k+1}) = \vec{t}$ . Letting

$$\Sigma_{\vec{t}} = \{(x^1, \dots, x^{k+1}) \in (\mathbb{R}^d)^{k+1} : \Phi(x^1, \dots, x^{k+1}) = \vec{t}\},$$

the continuous variant of the adjacency operator for this hyper-graph is given by

$$(1.5) \quad \mathcal{R}_k^d(f_1, \dots, f_k)(x^{k+1}) = \int_{\Sigma_{\vec{t}}^{x^{k+1}}} \prod_{j=1}^k f_j(x^j) d\sigma_{\vec{t}}^{x^{k+1}}(x^1, \dots, x^k),$$

where  $d\sigma_{\vec{t}}^{x^{k+1}}(x^1, \dots, x^k)$  is the Leray measure on the set

$$\Sigma_{\vec{t}}^{x^{k+1}} = \{(x^1, \dots, x^k) \in (\mathbb{R}^d)^k : \Phi(x^1, \dots, x^{k+1}) = \vec{t}\}.$$

This may be modified in an inessential way by multiplying  $d\sigma_{\vec{t}}^{x^{k+1}}$  by a smooth cut-off function  $\psi$ .

A model  $k$ -linear generalized Radon transform is the multilinear analogue of the linear spherical averaging operator. Taking advantage of its translation invariance, we define this as

$$(1.6) \quad A_k^d(f_1, \dots, f_k)(x) = \int \dots \int \prod_{j=1}^k f_j(x - u^j) dM_k^d(u^1, \dots, u^k),$$

where, for  $k \leq d$ ,  $dM_k^d$  is the Leray measure on the set

$$\Sigma_d^k = \{(u^1, \dots, u^k) \in S^{d-1} \times \dots \times S^{d-1} : |u^i - u^j| = 1; 1 \leq i < j \leq k\}.$$

Just as the spherical averaging operator arose naturally in Falconer's and subsequent investigations of the distance problem, the  $k$ -linear operator  $A_k^d$  arises naturally when considering the  $(k+1)$ -point configuration problem investigated in [10, 6, 25]. The question there was to determine how large the Hausdorff dimension of a subset of  $\mathbb{R}^d$  needs to be to ensure that the  $\binom{k+1}{2}$ -dimensional Lebesgue measure of non-congruent  $k$ -simplices (i.e.,  $(k+1)$ -point configurations) is positive.

The operator  $A_k^d$  and translation-invariant multilinear generalized Radon transforms in general can be put into context via a result of the first listed author and Soria [14] (stated there for bilinear convolution operators, but easily extended to multilinear ones), and an immediate consequence.

**THEOREM 1.1.** (*Grafakos and Soria 2010*) *Let  $\mu$  be a non-negative Borel measure on  $(\mathbb{R}^d)^k$  and set*

$$T_\mu(f_1, \dots, f_k)(x) = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} f_1(x - u^1) \dots f_k(x - u^k) d\mu(u^1, \dots, u^k).$$

*Suppose that  $\frac{1}{p_1} + \dots + \frac{1}{p_k} = \frac{1}{r} \leq 1$ . Then*

$$T_\mu : L^{p_1}(\mathbb{R}^d) \times \dots \times L^{p_k}(\mathbb{R}^d) \rightarrow L^r(\mathbb{R}^d)$$

*if and only if  $\mu$  is a finite measure. Furthermore, suppose that  $\gamma_j \in \mathbb{R}$  are such that the distribution  $(I - \Delta_{u_1})^{\frac{\gamma_1}{2}} \dots (I - \Delta_{u_k})^{\frac{\gamma_k}{2}} \mu$  is a Borel measure. Then for all  $p_j$  with  $\frac{1}{p_1} + \dots + \frac{1}{p_k} = \frac{1}{\tilde{r}} \leq 1$ ,*

$$T_\mu : L_{-\gamma_1}^{p_1}(\mathbb{R}^d) \times \dots \times L_{-\gamma_k}^{p_k}(\mathbb{R}^d) \rightarrow L^{\tilde{r}}(\mathbb{R}^d),$$

*where  $L_\gamma^p$  denotes the inhomogeneous Sobolev space of distributions  $g$  with  $(I - \Delta)^\gamma g \in L^p(\mathbb{R}^d)$ .*

The second part of Theorem 1.1 follows easily by expressing  $T_\mu$  in Fourier multiplier form

$$T_\mu(f_1, \dots, f_k)(x) = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \widehat{f}_1(\xi_1) \dots \widehat{f}_k(\xi_k) \widehat{\mu}(\xi_1, \dots, \xi_k) e^{2\pi i x \cdot (\xi_1 + \dots + \xi_k)} d\xi_1 \dots d\xi_k,$$

multiplying and dividing by  $(1+4\pi^2|\xi_1|^2)^{\gamma_1/2} \dots (1+4\pi^2|\xi_k|^2)^{\gamma_k/2}$ , taking inverse Fourier transforms, and using the first part of the theorem.

As a special case, consider the bilinear operator in the plane, i.e.  $k = d = 2$ , for which

$$(1.7) \quad A_2^2(f, g)(x) = \sum_{\pm} \int_0^{2\pi} f(x - (\cos(\theta), \sin(\theta))) g(x - (\cos(\theta \pm \pi/3), \sin(\theta \pm \pi/3))) d\theta.$$

In the notation of Theorem 1.1, the measure  $\mu$  is a multiple of arc-length on the curve

$$\{(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2 : |u| = |v| = |u - v| = 1\} \subset \mathbb{R}^4.$$

One can calculate [10] that  $\hat{\mu}(\xi, \eta) = \sum_{\pm} \hat{\sigma}(U_{\pm}(\xi, \eta))$ , where  $U_{\pm} : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  are the linear maps

$$U_{\pm}(\xi, \eta) = \left( \xi_1 + \frac{\eta_1}{2} \pm \eta_2 \frac{\sqrt{3}}{2}, \xi_2 \mp \eta_1 \frac{\sqrt{3}}{2} + \frac{1}{2} \eta_2 \right).$$

It follows that  $\hat{\mu} \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ , with no better uniform decay; however, the operator  $A_2^2$  satisfies much better bounds than those implied by Theorem 1.1. In fact, we shall see in the sequel that

$$(1.8) \quad A_2^2 : L_{-\frac{1}{2}}^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \rightarrow L^1(\mathbb{R}^2)$$

and, more generally,

$$(1.9) \quad A_2^d : L_{-\frac{d-1}{2}}^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d),$$

with corresponding non-trivial bounds for the operator  $A_k^d$ ,  $k \leq d$ .

In the first result of the current work, we establish certain bounds for multilinear generalized Radon transforms. Later, Theorem 5.1 and its corollaries will give results for nontranslation invariant multilinear generalized Radon transforms. However, all of our applications to continuous and discrete geometry are in fact made using the more restrictive class of translation invariant multilinear generalized Radon transforms and in this setting one can obtain stronger results.

**THEOREM 1.2.** *Let  $T_\mu$  be the multilinear convolution operator*

$$T_\mu(f_1, \dots, f_k)(x) = \int \dots \int f_1(x - u^1) \dots f_k(x - u^k) d\mu(u^1, \dots, u^k)$$

where  $\mu$  is a nonnegative Borel measure. Suppose that

$$(1.10) \quad |\hat{\mu}(-\xi, \xi, 0, \dots, 0)| \lesssim (1 + |\xi|)^{-\gamma}$$

for some  $\gamma > 0$ . Then, for all  $\gamma_1, \gamma_2 > 0$  such that  $\gamma = \gamma_1 + \gamma_2$ , and acting on nonnegative functions,

$$T_\mu : L_{-\gamma_1}^2(\mathbb{R}^d) \times L_{-\gamma_2}^2(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \times \dots \times L^\infty(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d).$$

Here, and throughout, the notation  $X \lesssim Y$  means that there exists a constant  $C > 0$ , independent of the variables of interest, such that  $X \leq CY$ .

In Theorem 1.2 (and Theorem 5.1 below) there is nothing special about the first two coordinates in the assumption of Fourier decay of the measure; one could of course state both theorems more generally for any two distinct coordinates, and correspondingly change the resulting boundedness conclusion. The key feature of our results is that they give non-trivial bounds for multilinear operators whose integral kernels are measures which have Fourier transform in  $L^\infty$  but satisfy no better uniform decay estimate. The positivity of the measures allows for the result to hold if the Fourier transform merely decays on the  $d$ -dimensional plane  $\eta = -\xi$ . Before treating a more general class of non-translation invariant operators, we give proofs and counterexamples related to the translation invariant theorems and discuss applications to continuous and discrete geometry.

The plan of the paper is as follows. In Sec. 2, we prove Theorem 1.2 and show that, despite its appearance, it is in some sense a bilinear theorem. In Sec. 3, we describe the a general framework

of variations on the Falconer distance problem for  $(k+1)$ -point configurations by means of what we call  $\Phi$ -configurations of points in  $E \subset \mathbb{R}^d$ , which include both  $k$ -simplices and their volumes. Applications of these results to Erdős-type problems in discrete geometry are given in Sec. 4. An extension of our main theorem to a nontranslation invariant setting and results about adjoints of multilinear operators are in Secs. 5 and 6, while Sec. 7 gives a version of the regular value theorem for sets of fractional dimension.

## 2. Translation invariant proof and its intrinsic bilinearity

**2.1. Proof of Theorem 1.2.** We assume that  $f_j \geq 0$  are Schwartz functions. Using  $f_j \geq 0$  in the first line and the assumption (1.10) for the first inequality, one sees that

$$\begin{aligned}
 \|T_\mu(f_1, \dots, f_k)\|_{L^1(\mathbb{R}^d)} &\leq \prod_{j=3}^k \|f_j\|_\infty \int \dots \int f_1(x - u^1) f_2(x - u^2) dx d\mu(u^1, \dots, u^k) \\
 &= \prod_{j=3}^k \|f_j\|_\infty \int \dots \int f_1(y) f_2(y + u^1 - u^2) dy d\mu(u^1, \dots, u^k) \\
 &= \prod_{j=3}^k \|f_j\|_\infty \int \dots \int f_1(y) \int \widehat{f}_2(\xi) e^{2\pi i \xi \cdot (y + u^1 - u^2)} d\xi dy d\mu(u^1, \dots, u^k) \\
 &= \prod_{j=3}^k \|f_j\|_\infty \int \widehat{f}_1(-\xi) \widehat{f}_2(\xi) \widehat{\mu}(-\xi, \xi, 0, \dots, 0) d\xi \\
 &\lesssim \prod_{j=3}^k \|f_j\|_\infty \int |\widehat{f}_1(-\xi)| |\widehat{f}_2(\xi)| (1 + |\xi|)^{-\gamma} d\xi \\
 &\lesssim \prod_{j=3}^k \|f_j\|_\infty \left( \int |\widehat{f}_1(\xi)|^2 (1 + |\xi|)^{-2\gamma_1} d\xi \right) \left( \int |\widehat{f}_2(\xi)|^2 (1 + |\xi|)^{-2\gamma_2} d\xi \right).
 \end{aligned}$$

**2.2. Bilinearity of the multilinear estimates.** Theorem 1.2 is inherently bilinear in nature: While the result is stated and used for multilinear operators, the assumption is distinctly bilinear in the sense that the number of derivatives gained over the trivial Holder estimate is based on the decay of the bilinear multiplier. We shall now see that this is essentially unavoidable in the sense that a translation invariant  $k$ -linear operator,  $k \geq 3$ , cannot in general gain the number of derivatives over the trivial Holder estimate corresponding to the optimal uniform decay of the  $k$ -linear multiplier. To see this, through a counterexample of geometric interest, let

$$T_d(f_1, \dots, f_d)(x) = \int \dots \int f_1(x - u^1) \dots f_d(x - u^d) d\Omega(u^1, \dots, u^d),$$

where, for any  $t \neq 0$ ,  $d\Omega$  is the Leray measure on the smooth determinantal variety

$$\Sigma_t = \{(u^1, \dots, u^d) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d : |u^1, \dots, u^d| := \det[u^1, \dots, u^d] = t\}.$$

**THEOREM 2.1.** *Let  $T_k$  be as above and suppose that  $d \geq 3$ . Then*

$$(2.1) \quad |\widehat{\Omega}(\xi^1, \dots, \xi^d)| \lesssim (1 + |\xi^1| + |\xi^2| + \dots + |\xi^d|)^{-\frac{d^2-1}{2}}.$$

However, suppose that  $p_j \geq 2$ ,  $\frac{1}{p_1} + \dots + \frac{1}{p_d} = 1$ ,  $\gamma_j \geq 0$  and  $\gamma_1 + \dots + \gamma_d = \frac{d^2-1}{2}$ . Then no estimate of the form

$$(2.2) \quad \|T_d(f_1, \dots, f_d)\|_{L^1(\mathbb{R}^d)} \lesssim \|f_1\|_{L_{-\gamma_1}^{p_1}(\mathbb{R}^d)} \times \dots \times \|f_d\|_{L_{-\gamma_d}^{p_d}(\mathbb{R}^d)}$$

can hold, even on nonnegative functions.

To start the proof of Theorem 2.1, we establish that the estimate (2.1) holds. Later we will prove that, if (2.2) were to hold, that would imply that if the Hausdorff dimension of a subset of  $\mathbb{R}^d$  is greater than a number of the form  $d - 1 - \epsilon$ , then the set of volumes determined by  $(k+1)$ -tuples of elements of  $E$  is positive. This is absurd, since  $E$  could be contained in a  $(d-1)$ -dimensional hyperplane.

To see (2.1), consider  $\Phi : \mathbb{R}^{d^2} \rightarrow \mathbb{R}$ ,  $\Phi(x^1, \dots, x^d) = |x^1, \dots, x^d|$ , so that  $\Sigma_t = \Phi^{-1}(t)$ . Then

$$(2.3) \quad d\Phi(X^1, \dots, X^d) = \sum_{i=1}^d |x^1, \dots, x^{i-1}, X^i, x^{i+1}, \dots, x^d|.$$

For a point  $(x^1, \dots, x^d) \in \Sigma_t$ , the vectors  $x^1, \dots, x^d$  are linearly independent in  $\mathbb{R}^d$ ; thus,  $d\Phi(x) \neq 0$ , since it is nonzero, e.g., when applied to vectors of the form  $(X^1, \dots, X^d) = (0, \dots, 0, c_i x^i, 0, \dots, 0)$ ,  $c_i \neq 0$ . Hence,  $\Sigma_t$  is a smooth hypersurface. Note also that  $SL(d, \mathbb{R})$  acts transitively on  $\Sigma_t$  through its diagonal action on  $\mathbb{R}^{d^2}$ , since if  $(x^1, \dots, x^d), (y^1, \dots, y^d) \in \Sigma_t$ , then there exists a (unique)  $A \in SL(\mathbb{R}, d)$  such that  $Ay^i = x^i$ ,  $1 \leq i \leq d$ . Thus, to show that  $\Sigma_t$  has nonzero Gaussian curvature everywhere, it suffices to consider its second fundamental form at a single point. Furthermore, since  $\Sigma_t$  is a homothetic copy of  $\Sigma_1$ , we may assume that  $t = 1$ . Thus, we may work at the point  $x_0 = (e^1, \dots, e^d) \in \Sigma_1$ , where the  $e^i$  are the standard orthonormal basis for  $\mathbb{R}^d$ .

Using the notation  $\vec{X} = (X^1, \dots, X^d) \in T\mathbb{R}^{d^2}$  and  $|\cdot, \cdot|_{ij}$  denoting the  $(i, j)$ -th  $2 \times 2$  minor of a  $d \times 2$  matrix, differentiating (2.3) again yields that

$$\begin{aligned} d^2\Phi(\vec{X}, \vec{X}) &= \sum_{1 \leq i < j \leq d} |X^i, X^j|_{ij}, \\ &= \sum_{1 \leq i < j \leq d} X_i^i X_j^j - X_j^i X_i^j \\ &= \sum_{1 \leq i < j \leq d-1} X_i^i X_j^j - X_j^i X_i^j + \sum_{i=1}^d X_i^i X_d^d - X_d^i X_i^d. \end{aligned}$$

Now restrict this quadratic form to  $T\Sigma_1$ , using

$$(2.4) \quad T_{x_0}\Sigma_1 = \left\{ (X^1, \dots, X^d) : X_1^1 + \dots + X_d^d = 0 \right\} = \left\{ X_d^d = -\sum_{i=1}^{d-1} X_i^i \right\};$$

this yields

$$(2.5) \quad \sum_{1 \leq i < j \leq d-1} X_i^i X_j^j - \sum_{1 \leq i < j \leq d-1} X_j^i X_i^j + \sum_{i=1}^{d-1} \left( X_i^i \left( -\sum_{j=1}^{d-1} X_j^j \right) \right) - \sum_{i=1}^{d-1} X_d^i X_i^d.$$

From the second and fourth terms, we see that for  $1 \leq i, j \leq d, i \neq j$ , the coordinate  $X_i^j$  occurs (only) in the term  $-X_j^i X_i^j$ ; each such term contributes 2 to the rank of the Hessian of  $\Phi$ . On the other hand, the  $d-1$  coordinates  $X_i^i$ , for  $1 \leq i \leq d-1$ , occur as  $-\sum_{1 \leq i \leq j \leq d-1} X_i^i X_j^j$ , and this quadratic form on  $\mathbb{R}^{d-1}$  is represented by a  $(d-1) \times (d-1)$  circulant matrix with first row  $[1, \frac{1}{2}, \dots, \frac{1}{2}]$ , which is easily seen to be nonsingular. The second fundamental form of  $\Sigma_1$  at  $x_0$  is thus nonsingular, showing that  $\Sigma_1$  has non-zero Gaussian curvature there, and hence everywhere. Hence, (2.1) follows from the standard stationary phase estimate (see, e.g., [39]).

We now prove that (2.2) cannot hold, using Theorem 3.3 below. If (2.2) were to hold, it would imply that if the Hausdorff dimension of  $E \subset \mathbb{R}^d$  is greater than  $d - \frac{\gamma}{d}$ , with  $\gamma = \frac{d^2-1}{2}$ , then the Lebesgue measure of  $\mathcal{V}_d(E)$  has positive Lebesgue measure. This is not in general possible, since

$$d - \frac{\gamma}{d} = k - \frac{d^2-1}{2k} = \frac{d}{2} + \frac{1}{2d},$$

and this is smaller than  $d-1$  when  $d \geq 3$ ; since  $E$  could be contained in a  $(d-1)$ -dimensional linear subspace, resulting in  $\mathcal{V}_d(E) = \{0\}$ , this is a contradiction. This proves Theorem 2.1.

### 3. Applications to problems in geometric measure theory

The classical Falconer distance problem, introduced in [8], can be stated as follows: How large does the Hausdorff dimension of  $E$  need to be to ensure that the Euclidian distance set  $\Delta(E) = \{|x-y| : x, y \in E\} \subset \mathbb{R}$  has positive one-dimensional Lebesgue measure? This problem can be viewed as a continuous analogue of the Erdős distance problem (see [30] and the references there). It is shown in [8], using the set obtained by a suitable scaling of the thickened integer lattice that the best result we can hope for is the following.

**CONJECTURE 3.1.** (*Falconer distance conjecture*) *Let  $E \subset \mathbb{R}^d$  with  $\dim_{\mathcal{H}}(E) > \frac{d}{2}$ . Then the one-dimensional Lebesgue measure  $\mathcal{L}^1(\Delta(E)) > 0$ .*

The best partial results known, due to Wolff [43] in the plane and to Erdoğan [5] in higher dimensions, say that  $\mathcal{L}^1(\Delta(E))$  is indeed positive if the Hausdorff dimension  $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{3}$ . The proofs are Fourier analytic in nature and rely, at least in higher dimensions, on bilinear extension estimates.

A multi-point configuration version of the Falconer distance problem is the following. Let  $E \subset \mathbb{R}^d$ ,  $d \geq 2$ , be compact; we may, for the sake of convenience, assume that  $E \subset [0, 1]^d$ . For  $2 \leq k \leq d+1$ , call two  $(k+1)$ -tuples of vectors from the set  $E \subset \mathbb{R}^d$  *equivalent* if there exists a rigid motion that maps one  $(k+1)$ -tuple to the other, and let  $T_k(E)$  denote the resulting set of equivalence classes. For  $k=1$ ,  $T_k(E) = \Delta(E)$ , the distance set defined above, while for  $k > 1$ ,  $T_k(E)$  can be thought of as the set of non-congruent  $(k+1)$ -point configurations. Note that, since rigid motions preserve distances,  $T_k(E)$  can be naturally realized as a subset of  $\mathbb{R}^{\binom{k+1}{2}}$  and thus the property of  $T_k(E)$  having positive Lebesgue measure,  $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$ , is well-defined. It is then reasonable to ask whether one can find  $0 < s_0 < d$  such that if  $\dim_{\mathcal{H}}(E) > s_0$ , then  $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$ . Relatedly, one can ask “how likely” it is that a given configuration can arise.

More generally, if

$$(3.1) \quad \Phi : (\mathbb{R}^d)^{k+1} \rightarrow \mathbb{R}^m,$$

for some  $1 \leq m \leq \binom{d+1}{2}$ , we define the set of  $\Phi$ -configurations of  $E$ ,

$$\Delta_\Phi(E) = \{\Phi(x^1, \dots, x^{k+1}) : x^j \in E\}.$$

In analogy with the questions above, we ask how large the Hausdorff dimension of  $E$  needs to be to ensure that  $\mathcal{L}^m(\Delta_\Phi(E)) > 0$ . Letting  $\nu$  be a Frostman measure supported on  $E$ , we also ask how likely it is for  $\Phi$ -configurations to be near a fixed one by asking how large the Hausdorff dimension of  $E$  needs to be to ensure that uniform estimates of the form

$$(3.2) \quad (\nu \times \dots \times \nu)\{(x^1, \dots, x^{k+1}) : |\Phi(x^1, \dots, x^{k+1}) - \bar{t}| < \epsilon\} \lesssim \epsilon^m$$

hold. We will focus on the translation invariant case, for which  $\Phi$  can be written as

$$\Phi(x^{k+1} - x^1, \dots, x^{k+1} - x^k).$$

Our main result is the following.

**THEOREM 3.2.** *Let  $\Phi$  be translation invariant,  $\Delta_\Phi(E)$  be defined as above, and  $\mu_t$  be the Leray measure on*

$$\{(u^1, \dots, u^k) : \Phi(u^1, \dots, u^k) = \bar{t}\}$$

*(assumed to be smooth). Suppose that there exists a  $\gamma > 0$  such that for all ordered  $k$ -tuples  $\Xi_{j,\ell}$ ,  $j \neq \ell$ , with  $-\xi$  in the  $j$ th entry and  $\xi$  in the  $\ell$ th entry and 0 in the remaining ones,*

$$|\hat{\mu}(\Xi_{j,\ell})| \lesssim (1 + |\xi|)^{-\gamma},$$

*and that for all ordered  $k$ -tuples  $\Xi_i$  with  $\xi$  in the  $i$ th entry and 0 in the remaining entries,*

$$|\hat{\mu}(\Xi_i)| \lesssim (1 + |\xi|)^{-\gamma}.$$

*Assume further that the Hausdorff dimension of  $E$  is greater than  $d - \frac{\gamma}{k}$ . Then (3.2) holds and the  $m$ -dimensional Lebesgue measure  $\mathcal{L}^m(\Delta_\Phi(E)) > 0$ .*

Theorem 3.2 follows directly from Theorem 1.2, Theorem 6.1 and the following result.

**THEOREM 3.3.** *Let  $\Phi$  and  $\mu_t$  be defined as above and define*

$$T_\Phi(f_1, \dots, f_k)(x) = \int \dots \int f_1(x - u_1) \dots f_k(x - u_k) d\mu_t(u_1, \dots, u_k).$$

*Suppose that  $p_j \geq 2$ ,  $\frac{1}{p_1} + \dots + \frac{1}{p_k} = 1$ ,  $\gamma_j \geq 0$ ,  $\gamma_1 + \dots + \gamma_k = \gamma$ , and*

$$(3.3) \quad \|T_\Phi(f_1, \dots, f_k)\|_{L^1(\mathbb{R}^d)} \lesssim \|f_1\|_{L^{p_{\sigma(1)}}_{-\gamma_{\sigma(1)}}(\mathbb{R}^d)} \dots \|f_k\|_{L^{p_{\sigma(k)}}_{-\gamma_{\sigma(k)}}(\mathbb{R}^d)}$$

*for all permutations  $\sigma$  and that the same estimates hold for each multilinear adjoint of  $T_\Phi$ . Then, if  $\dim_{\mathcal{H}}(E) > d - \frac{\gamma}{k}$ , (3.2) holds and hence  $\mathcal{L}^m(\Delta_\Phi(E)) > 0$ .*

**3.1. Distribution of simplices.** Our main result on the distribution of simplices in fractal sets is the following. This result was previously established [10] in the case  $d = k = 2$ .

**THEOREM 3.4.** *Suppose that the Hausdorff dimension of a compact set  $E \subset \mathbb{R}^d$  is greater than  $s_0(k, d) = d - \frac{d-1}{2k}$ ,  $1 \leq k \leq d$ ,  $\nu$  is a Frostman measure supported on  $E$ , and  $\{t_{ij}\}_{1 \leq i < j \leq k+1}$  a collection of positive real numbers. Then*

$$(\nu \times \dots \times \nu)\{(x^1, \dots, x^{k+1}) : t_{ij} - \epsilon \leq |x^i - x^j| \leq t_{ij} + \epsilon\} \lesssim \epsilon^{\binom{k+1}{2}}$$

*with constants independent of  $\epsilon$ . Consequently,*

$$(3.4) \quad \mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0 \text{ if } \dim_{\mathcal{H}}(E) > d - \frac{d-1}{2k}.$$



REMARK 3.5. When  $k = 1$ , Theorem 3.4 is known to be essentially sharp. See, e.g., [24] and the references there. When  $k = d = 2$ , the estimate is also sharp, as was shown in [10]. In all the remaining cases, we believe that the Hausdorff exponents can be improved, and hope to address this in the future.

REMARK 3.6. It is proved in [6] that  $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$  if  $\dim_{\mathcal{H}}(E) > \frac{d+k+1}{2}$ . One can check that the exponent in (3.4) is better when  $d < k + 2 + \frac{1}{k-1}$ . Moreover, the exponent  $\frac{d+k+1}{2}$  is only  $< d$  when  $k < d - 1$ , whereas the exponent in (3.4) is always non-trivial.

**3.2. Distribution of volumes of simplices.** In [6], Erdoğan, Hart and the third listed author proved that if the Hausdorff dimension of  $E \subset [0, 1]^d$  is greater than  $\frac{d+1}{2}$ , then the Lebesgue measure of the set of areas determined by triangles formed by pairs of points from  $E$  and the origin is positive. In [11], Mouroglou, the second and the third listed authors proved an analogous result for volumes of simplices in  $\mathbb{R}^3$  determined by three points of a given set and the origin. They prove that if  $E \subset \mathbb{R}^3$  and  $\dim_{\mathcal{H}}(E) > 8/3$ , then the resulting set of volumes has positive Lebesgue measure in  $\mathbb{R}$ .

In this section, we obtain a better exponent than the one in [6] by considering  $k$ -dimensional volumes of simplices determined by  $k + 1$  points in  $E$ ; a bilinear point of view once again plays a crucial role. Our main result is the following.

THEOREM 3.7. *Define  $\mathcal{V}_d(E)$  to be set of  $d$ -dimensional volumes determined by  $(d + 1)$ -tuples of points from  $E \subset [0, 1]^d$ ,  $d \geq 2$ . Suppose that  $\dim_{\mathcal{H}}(E)$  is greater than  $d - 1 + \frac{1}{2d}$  if  $d$  is even, and greater than  $d - 1 + \frac{1}{2(d-1)}$  if  $d$  is odd. Then the Lebesgue measure of  $\mathcal{L}^1(\mathcal{V}_d(E)) > 0$ .*

We shall prove this result in even dimensions and bootstrap it into odd dimensions using the following mechanism which is interesting in its own right.

THEOREM 3.8. *Suppose that, whenever Hausdorff dimension of  $E_{d-1} \subset \mathbb{R}^{d-1}$ ,  $d \geq 3$ , is greater than  $s_{d-1} \in (d - 2, d - 1)$ , then the Lebesgue measure of  $\mathcal{V}_{d-1}(E_{d-1})$  is positive. Then, if the Hausdorff dimension of  $E_d \subset \mathbb{R}^d$  is greater than  $s_d = s_{d-1} + 1$ , the Lebesgue measure of  $\mathcal{L}^1(\mathcal{V}_d(E_d)) > 0$ .*

REMARK 3.9. One immediate implication of Theorem 3.8 is that if we could prove that  $s_2 = 1$ , then  $s_d = d - 1$ , which would be the sharp exponent. Our current best exponent in two dimensions is  $5/4$ , which, in view of Theorem 3.8 leads to the exponent  $d - \frac{3}{4}$  in  $\mathbb{R}^d$ . We are able to obtain a better estimate in higher dimensions using multilinear operator bounds. Nevertheless, it would be reasonable to suppose that the ultimate resolution of the sharp exponent  $d - 1$  in  $d$ -dimensions will ultimately be accomplished by proving the sharp bound in dimension two and then using the bootstrapping mechanism of Theorem 3.8.

REMARK 3.10. With a bit of work, our method extends to  $k$ -dimensional volumes determined by  $(k + 1)$ -tuples of vectors in  $\mathbb{R}^d$ . One can check that the main technical adjustment comes in the pigeon-holing argument at the end of the proof of Theorem 3.7. In particular, the dimensional threshold exponent for 2-dimensional volumes in  $\mathbb{R}^d$  is

$$d - \frac{d - \frac{1}{2}}{2} = \frac{d}{2} + \frac{1}{4}.$$

The details are left to the interested reader.

**3.3. Distribution of angles.** The following result is proved in [25] and is being included to illustrate the range of problems to which our method applies.

DEFINITION 3.11. Let  $E \subset \mathbb{R}^d$ ,  $d \geq 2$ . We say that an angle  $\alpha \in [0, \pi]$  is *equitably represented* in  $\mathcal{A}(E)$  if for every Frostman measure  $\mu$  supported on  $E$  and any  $\epsilon > 0$ ,

$$(3.5) \quad (\mu \times \mu \times \mu)\{(x^1, x^2, x^3) : \alpha - \epsilon \leq \theta(x^1, x^2, x^3) \leq \alpha + \epsilon\} \lesssim \epsilon.$$

THEOREM 3.12. Let  $E$  be a compact subset of  $\mathbb{R}^d$  of Hausdorff dimension greater than  $\frac{d+1}{2}$ . Then every  $\alpha \in [0, \pi]$  is equitably represented in  $\mathcal{A}(E)$ .

COROLLARY 3.13. Let  $E$  be a compact subset of  $\mathbb{R}^d$  of Hausdorff dimension greater than  $\frac{d+1}{2}$ . Then the Lebesgue measure of  $\mathcal{A}(E)$  is positive.

Theorem 3.12 can be recovered from Theorem 3.2 via the following lemma proved in [25].

LEMMA 3.14. Let

$$(3.6) \quad \widehat{\mu}_t(\xi, \eta) = \iiint e^{-2\pi i(u \cdot \xi + \lambda \theta u \cdot \eta)} \psi(|u|) \psi_0(\lambda) d\Omega_{t, \frac{u}{|u|}}(\theta) du d\lambda,$$

where  $d\Omega_{t, \frac{u}{|u|}}(\theta)$  is the restriction of the Haar measure on  $SO(d)$  to  $\Omega_{t, \frac{u}{|u|}}$  and  $\psi, \psi_0$  are smooth cut-off functions. Then

$$|\widehat{\mu}_t(\xi, \eta)| \lesssim \frac{1}{(1 + |\xi| + |\eta|)^{d-1}}.$$

**3.4. Proof of Theorem 3.3.** Let  $\nu$  be a Frostman measure, supported on  $E \subset \mathbb{R}^d$ . Set  $s = \dim_{\mathcal{H}}(E)$ . Let  $\rho$  be a non-negative, smooth function, equal to 1 on  $[-\frac{1}{4}, \frac{1}{4}]$ , supported in  $[-1, 1]$  with  $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$ , and  $\nu^\delta := \nu * \rho_\delta$ , where  $\rho_\delta(x) = \delta^{-d} \rho(\frac{x}{\delta})$ , the resulting smooth approximation of  $\nu$  as  $\delta$  tends to 0. We will establish the bound

$$\frac{1}{\epsilon^m} (\nu^\delta \times \dots \times \nu^\delta)\{(x^1, \dots, x^{k+1}) : |\Phi(x^{k+1} - x^1, \dots, x^{k+1} - x^k) - \vec{t}| < \epsilon\} \lesssim 1,$$

uniformly in  $\delta$ , and thus by passing to the limit we establish the theorem. Write

$$\begin{aligned} & \frac{1}{\epsilon^m} (\nu^\delta \times \dots \times \nu^\delta)\{(x^1, \dots, x^{k+1}) : |\Phi(x^{k+1} - x^1, \dots, x^{k+1} - x^k) - \vec{t}| < \epsilon\} \\ &= \epsilon^{-m} \int \dots \int_{\{(x^1, \dots, x^{k+1}) : |\Phi(x^{k+1} - x^1, \dots, x^{k+1} - x^k) - \vec{t}| < \epsilon\}} d\nu^\delta(x_1) \dots d\nu^\delta(x_k) d\nu^\delta(x_{k+1}) \\ &= \epsilon^{-m} \int \dots \int_{\{(u^1, \dots, u^{k+1}) : |\Phi(u^1, \dots, u^k) - \vec{t}| < \epsilon\}} d\nu^\delta(u_1) \dots d\nu^\delta(u_k) d\nu^\delta(x_{k+1}) \\ (3.7) \quad &= \langle T_\Phi(\nu^\delta, \dots, \nu^\delta), \nu^\delta \rangle, \end{aligned}$$

and  $\langle \cdot, \cdot \rangle$  is the  $L^2(\mathbb{R}^d)$  inner product.

Now define

$$F(\alpha_1, \dots, \alpha_k, \alpha_{k+1}) := \langle T_\Phi(\nu_{\alpha_1}^\delta, \dots, \nu_{\alpha_k}^\delta), \nu_{\alpha_{k+1}}^\delta \rangle$$

where, initially defined for  $\text{Re}(\beta) > 0$ ,

$$(3.8) \quad \nu_\beta^\delta(x) := \frac{\pi^{\frac{d-\beta}{2}}}{\Gamma(\beta/2)} (\nu^\delta * |\cdot|^{-d+\beta})(x)$$

is extended to the complex plane by analytic continuation. Since  $\nu_\beta^\delta$  is smooth and we are in a compact setting, we have trivial bounds on  $F(\alpha_1, \dots, \alpha_{k+1})$  with constants depending on  $\delta$ . Observe that  $\widehat{\nu_\beta^\delta}(\xi) = C_{\beta,d} \widehat{\nu}(\xi) \widehat{\rho}(\delta\xi) |\xi|^{-\beta}$  where

$$(3.9) \quad C_{\beta,d} = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d-\beta}{2})}.$$

See, e.g., [9, p. 192] for this and related calculations. By Plancherel,  $\nu_\beta^\delta$  is an  $L^2(\mathbb{R}^d)$  function with bounds depending on  $\delta$ . Taking the modulus in (3.8), we see that

$$|\mu_\beta^\delta(x)| \leq \left| \frac{\pi^{\frac{d-\beta}{2}}}{\Gamma(\frac{\beta}{2})} \right| (\nu^\delta * |\cdot|^{-d+\text{Re}(\beta)})(x) = \pi^{\frac{d-\text{Re}(\beta)}{2}} \frac{\Gamma(\text{Re}(\frac{\beta}{2}))}{|\Gamma(\frac{\beta}{2})|} \nu_{\text{Re}(\beta)}^\delta(x)$$

and note, using (3.8), that the right hand side is non-negative. Theorem 3.3 would thus follow if we could show that, whenever  $\dim_{\mathcal{H}}(E) > d - \frac{\gamma}{k}$ , then  $F(0, \dots, 0) \lesssim 1$ .

We shall bound  $F(\alpha_1, \dots, \alpha_{k+1})$ , with a constant independent of  $\delta$ , for every possible  $(k+1)$ -tuple of complex numbers where

$$(3.10) \quad \text{Re}(\alpha_i) = \frac{\gamma}{kp'_i} - \gamma_i$$

for  $1 \leq i \leq k$  and

$$(3.11) \quad \text{Re}(\alpha_{k+1}) = \frac{\gamma}{k}$$

and all permutations of such numbers. By the three lines lemma we then obtain a bound for  $F(\alpha_1, \dots, \alpha_{k+1})$ , independent of  $\delta$ , in the convex hull of these numbers. It is not difficult to see that the origin in  $\mathbb{C}^{k+1}$  is contained in the convex hull as the sum of the real parts of these numbers is 0 and, for each coordinate, all the  $\alpha_i$ 's appear equally often.

We get boundedness for each permutation of the  $\alpha$ 's from precisely one assumption in the statement of the theorem. Without loss of generality,  $\alpha_i$  equals the expression in (3.10) for  $1 \leq i \leq k$  and  $\alpha_{k+1} = \frac{\gamma}{k}$ . We have

$$(3.12) \quad F(\alpha_1, \dots, \alpha_{k+1}) = \langle T(\nu_{\alpha_1}^\delta, \nu_{\alpha_2}^\delta, \dots, \nu_{\alpha_k}^\delta), \nu_{\alpha_{k+1}}^\delta \rangle.$$

We need the following simple observation.

LEMMA 3.15. *If  $\nu$  is a Frostman measure on a set  $E \subset \mathbb{R}^d$  with  $\dim_{\mathcal{H}}(E) > s$ , then*

$$\|\nu_\alpha^\delta\|_\infty \lesssim 1 \text{ if } \text{Re}(\alpha) = d - s.$$

To prove the lemma, observe that if  $\text{Re}(\alpha) = d - s$ ,

$$|\nu_\alpha^\delta(x)| \leq \int |x - y|^{-s} d\nu^\delta(y) \approx \sum_j 2^{js} \int_{|x-y| \approx 2^{-j}} d\nu^\delta(y) \lesssim \sum_j 2^{js} 2^{-j \cdot \dim_{\mathcal{H}}(E)},$$

and this is  $\lesssim 1$ , since  $\nu$  is a Frostman measure on  $E$  and  $\dim_{\mathcal{H}}(E) > s$ .

We now bound

$$(3.13) \quad \begin{aligned} |F(\alpha)| &\leq \|T(\nu_{\alpha_1}^\delta, \nu_{\alpha_2}^\delta, \dots, \nu_{\alpha_k}^\delta)\|_1 \|\nu_{\alpha_{k+1}}^\delta\|_\infty \\ &\lesssim \|T(\nu_{\text{Re}(\alpha_1)}^\delta, \nu_{\text{Re}(\alpha_2)}^\delta, \dots, \nu_{\text{Re}(\alpha_k)}^\delta)\|_1 \|\nu_{\text{Re}(\alpha_{k+1})}^\delta\|_\infty, \end{aligned}$$

where the implicit constant depends on terms with gamma functions. Note that if, e.g.,  $\frac{\gamma}{2k} \in \mathbb{Z}$ , then our bounds blow up. This is however not an obstacle, because we can instead consider  $\frac{\gamma}{2k} + \epsilon$ , where  $\epsilon > 0$  is chosen so small that we still have that  $\dim_{\mathcal{H}}(E) > d - \frac{\gamma}{k} + \epsilon$ .

By Lemma 3.15, the expression in (3.13) is bounded by

$$(3.14) \quad \|T(\nu_{\text{Re}(\alpha_1)}^\delta, \nu_{\text{Re}(\alpha_2)}^\delta, \dots, \nu_{\text{Re}(\alpha_k)}^\delta)\|_1.$$

By assumption this expression is bounded by

$$(3.15) \quad \prod_{j=1}^k \|\nu_{\text{Re}(\alpha_j)}^\delta\|_{L_{-\gamma_j}^{p_j}(\mathbb{R}^d)}$$

and all we need to do is to establish the boundedness of  $\|\nu_{\text{Re}(\alpha_j)}^\delta\|_{L_{-\gamma_j}^{p_j}(\mathbb{R}^d)}$  for each  $j$ . Note that we can rewrite this as

$$\|\nu_{\text{Re}(\alpha_j) + \gamma_j}^\delta\|_{L^{p_j}(\mathbb{R}^d)} = \|\nu_{\frac{\gamma}{k}(1 - \frac{1}{p_j})}^\delta\|_{L^{p_j}(\mathbb{R}^d)}.$$

We now calculate using Hölder

$$\begin{aligned} \|\nu_{\frac{\gamma}{k}(1 - \frac{1}{p_j})}^\delta\|_{L^{p_j}(\mathbb{R}^d)}^{p_j} &\lesssim \int \left( \nu^\delta(y) |x - y|^{-d + \frac{\gamma}{k}(1 - \frac{1}{p_j})} \right)^{p_j} dx \\ &= \int \left( \int \left( \nu^\delta(y) |x - y|^{-d + \frac{\gamma}{k}} \right)^{1 - \frac{2}{p_j}} \left( \nu^\delta(y) |x - y|^{-d + \frac{\gamma}{2k}} \right)^{\frac{2}{p_j}} dy \right)^{p_j} dx \\ &\lesssim \int \left( \int \nu^\delta(y) |x - y|^{-d + \frac{\gamma}{k}} dy \right)^{p_j - 2} \left( \int \nu^\delta(y) |x - y|^{-d + \frac{\gamma}{2k}} dy \right)^2 dx \\ &\lesssim \|\nu_{\frac{\gamma}{k}}^\delta\|_{L^\infty(\mathbb{R}^d)}^{p_j - 2} \|\nu_{\frac{\gamma}{2k}}^\delta\|_{L^2(\mathbb{R}^d)}^2 \end{aligned}$$

and this is bounded using Lemma 3.15 and the following lemma.

LEMMA 3.16. *If  $\nu$  is a Frostman measure on a set  $E \subset \mathbb{R}^d$  with  $\dim_{\mathcal{H}}(E) > s$ , then*

$$\|\nu_\alpha^\delta\|_{L^2(\mathbb{R}^d)} \lesssim 1 \text{ if } \text{Re}(\alpha) = \frac{s}{2}.$$

To prove the lemma, first recall the energy integral,

$$I_s(\nu) = \int \int |x - y|^{-s} d\nu(x) d\nu(y).$$

A standard calculation shows

$$\|\nu_\alpha^\delta\|_{L^2(\mathbb{R}^d)} \lesssim (I_{d-2\text{Re}(\alpha)}(\nu^\delta))^{1/2} = (I_{d-s}(\nu^\delta))^{1/2}$$

and since  $\nu$  is supported on a set of Hausdorff dimension greater than  $s$ , we can bound  $I_{d-s}(\nu^\delta) \lesssim 1$ , with a bound independent of  $\delta$ .

**3.5. Proof of Theorem 3.4.** In light of Theorem 3.2 it suffices to establish the following multiplier estimate:

LEMMA 3.17. *On  $\mathbb{R}^d \times \mathbb{R}^d$ ,  $d \geq 3$ , let*

$$K(x, y) = \delta(|x| - 1) \delta(|y| - 1) \delta(|x - y| - 1) \simeq \delta(|x|^2 - 1) \delta(|y|^2 - 1) \delta(x \cdot y - \frac{1}{2}).$$

Then

$$|\widehat{K}(\xi, \eta)| \lesssim (1 + |\xi| + |\eta|)^{-\frac{d-1}{2}}.$$

To prove the lemma, we use a partition of unity to decompose  $K$  into a finite sum of terms supported on products of small spherical caps about points  $x^j, y^j$ ,  $1 \leq j \leq N_d$ . By rotation invariance of  $K$ , we can assume that the basepoints are  $x^0 = (0, \mathbf{0}', 1)$  and  $y^0 = (\frac{\sqrt{3}}{2}, \mathbf{0}', \frac{1}{2})$ , where we write  $\mathbb{R}^d \ni x = (x_1, x', x_d)$ . Introduce local coordinates on  $\mathbb{S}^{d-1}$  near  $x^0, y^0$ , resp.,

$$\begin{aligned} x(u) &= (u, 1 - \frac{|u|^2}{2}) + \mathcal{O}(|u|^3), \quad u = (u_1, u') \in \mathbb{R}^{d-1}, \quad |u| \leq \epsilon, \quad \text{and} \\ y(v) &= (\frac{\sqrt{3}}{2} + v_1, v', \frac{1}{2} - \sqrt{3}v_1 - 4v_1^2 - |v'|^2) + \mathcal{O}(|v|^3), \quad v = (v_1, v') \in \mathbb{R}^{d-1}, \quad |v| < \epsilon. \end{aligned}$$

All calculations that follow will be modulo  $\mathcal{O}^3 := \mathcal{O}(|u, v|^3)$ .

The measure  $K^0$  is a smooth multiple of surface measure on the  $(2d-3)$ -dimensional manifold  $\{(u, v) : x(u) \cdot y(v) = \frac{1}{2}\} \subset \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$ , pushed forward under the parametrization map  $(u, v) \rightarrow (x(u), y(v))$ . However,

$$\begin{aligned} x(u) \cdot y(v) - \frac{1}{2} &= \frac{\sqrt{3}}{2}u_1 - \sqrt{3}v_1 + u \cdot v - \frac{|u|^2}{4} - 4v_1^2 - |v'|^2 + \mathcal{O}^3 = 0 \Leftrightarrow \\ (\sqrt{3} - u_1)v_1 + 4v_1^2 &= \frac{\sqrt{3}}{2}u_1 + u' \cdot v' - \frac{|u|^2}{4} - |v'|^2 + \mathcal{O}^3. \end{aligned}$$

We use the quadratic terms in the implicit function theorem in one variable,

$$(3.16) \quad a_1 s + a_2 s^2 = t \implies s = a_1^{-1}t - a_1^{-3}a_2 t^2 + \mathcal{O}(t^3), \quad s, t \searrow 0,$$

to solve for  $v_1$  in terms of  $u_1$ , with  $u', v'$  as parameters:

$$(3.17) \quad v_1 = \frac{1}{2}u_1 - \frac{13\sqrt{3}}{36}u_1^2 - \frac{\sqrt{3}}{12}|u'|^2 - \frac{1}{\sqrt{3}}|v'|^2 + \frac{1}{\sqrt{3}}u' \cdot v' + \mathcal{O}^3,$$

so that also  $v_1^2 = \frac{1}{4}u_1^2 + \mathcal{O}^3$ . Hence, the contribution  $dK^0$  of this pair of spherical caps to  $dK$  is the pushforward of  $du_1 du' dv'$  under

$$\begin{aligned} (u_1, u', v') &\rightarrow (u_1, u', 1 - \frac{1}{2}u_1^2 - \frac{1}{2}|u'|^2; \\ &\quad \frac{\sqrt{3}}{2} + \frac{1}{2}u_1 - \frac{13\sqrt{3}}{36}u_1^2 - \frac{\sqrt{3}}{12}|u'|^2 - \frac{1}{\sqrt{3}}u' \cdot v', \\ &\quad v', \frac{1}{2} - \frac{\sqrt{3}}{2}u_1 + \frac{7}{6}u_1^2 + \frac{1}{4}|u'|^2 - u' \cdot v'). \end{aligned}$$

Thus, for  $\chi$  a suitable smooth cutoff supported near  $(\mathbf{0}, \mathbf{0}')$ ,

$$(3.18) \quad \widehat{K}^0(\xi, \eta) = \int e^{-i\Phi(u_1, u', v')} \chi(u_1, u', v') du_1 du' dv',$$

where the phase function is

$$\begin{aligned}\Psi &= \xi_1 u_1 + \xi' \cdot u' + \xi_d \left(1 - \frac{|u|^2}{2}\right) + \eta' \cdot v' \\ &\quad + \eta_1 \left(\frac{\sqrt{3}}{2} + \frac{1}{2}u_1 - \frac{13\sqrt{3}}{36}u_1^2 - \frac{\sqrt{3}}{12}|u'|^2 - \frac{1}{\sqrt{3}}|v'|^2 + \frac{1}{\sqrt{3}}u' \cdot v'\right) \\ &\quad + \eta_d \left(\frac{1}{2} - \frac{\sqrt{3}}{2}u_1 + \frac{7}{6}u_1^2 + \frac{1}{4}|u'|^2 - u' \cdot v'\right),\end{aligned}$$

and the (linear) dependence of  $\Psi$  on  $(\xi, \eta)$  is notationally suppressed. For  $(\xi, \eta)$  fixed,  $\Psi$  has a unique critical point; conversely, if we fix  $(u, v')$ , then the system of linear equations

$$(3.19) \quad d_{u_1}\Psi(u, v') = 0, \quad d_{u'}\Psi(u, v') = d_{v'}\Psi(u, v') = 0$$

in  $(\xi, \eta)$  is of maximal rank, with a 3-dimensional solution space, corresponding to the conormal bundle of  $\Sigma$  at  $(x(u), y(v_1(u, v'), v'))$ . Without loss of generality, we consider  $(u_1, u', v') = (0, \mathbf{0}', \mathbf{0}')$  and show the claimed decay rate for  $\widehat{K^0}$  in the conormal plane there,  $N^*\Sigma = \{\xi_1 + \eta_1 - \frac{\sqrt{3}}{2}\eta_d = 0, \xi' = \eta' = 0\}$ . Splitting off  $u_1$  and then pairing  $u_i, v_i$ ,  $2 \leq i \leq d-1$ , one computes that the Hessian of  $\Psi$  at  $(0, \mathbf{0}', \mathbf{0}')$  is

$$(3.20) \quad p(\xi, \eta) \mathbb{I}_{1 \times 1} \oplus \bigoplus_2^{d-1} \begin{bmatrix} q_{11}(\xi, \eta) & q_{12}(\xi, \eta) \\ q_{21}(\xi, \eta) & q_{22}(\xi, \eta) \end{bmatrix},$$

where

$$(3.21) \quad p(\xi, \eta) = -\xi_d - \frac{13\sqrt{3}}{18}\eta_1 + \frac{7}{3}\eta_d$$

and

$$(3.22) \quad q_{11} = -(\xi_d + \frac{\sqrt{3}}{6}\eta_1 - \frac{1}{2}\eta_d), \quad q_{12} = q_{21} = \frac{1}{\sqrt{3}}\eta_1 - \eta_d, \quad q_{22} = -\frac{2}{\sqrt{3}}\eta_1.$$

This has determinant  $\pm p(\xi, \eta) \cdot q(\xi, \eta)^{d-2}$ , where

$$(3.23) \quad q(\xi, \eta) = \begin{vmatrix} q_{11}(\xi, \eta) & q_{12}(\xi, \eta) \\ q_{21}(\xi, \eta) & q_{22}(\xi, \eta) \end{vmatrix}.$$

If we restrict to the hyperplane  $\Pi := \{p = 0\}$  by setting  $\xi_d = -\frac{13\sqrt{3}}{18}\eta_1 + \frac{7}{3}\eta_d$ , then

$$(3.24) \quad q = -\frac{13}{9}\eta_1^2 + \frac{17\sqrt{3}}{9}\eta_1\eta_d - \eta_d^2,$$

which has negative discriminant and thus is a negative-definite quadratic form. Hence, within  $N^*\Sigma$ , the cone  $\Gamma := \{q(\xi, \eta) = 0\}$  intersects  $\Pi$  only at the origin. Off of  $\Gamma$ ,  $\text{rank } \Psi'' \geq 2(d-2)$ , so that stationary phase gives  $|\widehat{K^0}(\xi, \eta)| \lesssim (1 + |\xi| + |\eta|)^{-(d-2)}$ , which is, for  $d \geq 3$ , at least as good as in the statement of the proposition. On the other hand, by (3.23), (3.24) and the comment following it, the rank of  $\Pi$  is at least  $1 + (d-2) = d-1$ , yielding  $|\widehat{K^0}(\xi, \eta)| \lesssim (1 + |\xi| + |\eta|)^{-\frac{d-1}{2}}$ . By standard proofs of stationary phase, these estimates are uniform both in the conormal directions above the base point, and also as the base point is varied, completing the proof of Lemma 3.17.

**3.6. Proofs of Thms. 3.7 and 3.8.** We first prove Theorem 3.8. If  $E \subset \mathbb{R}^d$ ,  $d \geq 3$ , and  $\dim_{\mathcal{H}}(E) > s_d$ , then there exists a  $(d-1)$ -plane  $H$  such that  $\dim_{\mathcal{H}}(E \cap H) > s_d - 1$ ; see, e.g., [30]. Define  $s_{d-1} = s_d - 1$ . By assumption,  $\mathcal{V}_{d-1}(E \cap H) > 0$ . Let  $H'$  denote a  $(d-1)$ -plane parallel to  $H$ , but not equal to  $H$ , containing at least one point of  $E$ , denoted by  $z$ . Such a plane must exist since  $s_d > d-1$  by assumption. For every  $d$ -tuple in  $E \cap H$  that contributes a non-zero element to  $\mathcal{V}_{d-1}(E \cap H)$ , form a  $(d+1)$ -tuple in  $\mathbb{R}^d$  by adjoining  $z$ . The volumes of the resulting  $(d+1)$ -tuples are all distinct and form a set of positive one-dimensional Lebesgue measure. This completes the proof of Theorem 3.8.

To prove Theorem 3.7 we first observe that Theorem 3.2 goes over with no substantive changes if instead of assuming that each of vectors in the  $(k+1)$ -point configuration under consideration being contained in a single set  $E$ , we have  $k+1$  sets and  $x^j \in E_j$ ,  $j = 1, 2, \dots, k+1$ . The dimensional assumption then becomes  $\dim_{\mathcal{H}}(E_j) > d - \frac{\gamma}{k}$  for each  $1 \leq j \leq k+1$ .

Now, in view of Theorem 3.2, the even dimensional case of Theorem 3.7 would follow from:

LEMMA 3.18. *Let*

$$f(x, y) = \det \begin{pmatrix} x \\ y \\ z^1 \\ z^2 \\ \cdot \\ \cdot \\ \cdot \\ z^{d-2} \end{pmatrix},$$

where  $z^j \in E^j$  as above. Then for any ball  $B \subset \mathbb{R}^{2d}$  and  $t \neq 0$ , the hypersurface  $\{(x, y) : f(x, y) = t\}$  has at least  $2d-1$  non-vanishing principal curvatures, with bounds independent of  $z^j$ , and thus the Fourier transform of the Leray measure on this surface decays of order  $-\frac{2d-1}{2}$  at infinity.

To prove Lemma 3.18, consider the submatrix

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix},$$

When computing the determinant of the whole matrix, we get  $x_1y_2 - x_2y_1$  times the determinant of the matrix obtained by covering up the first two columns and the first two rows. The determinant of this matrix is non-zero by definition. Thus

$$f(x, y) = c_{12}(x_1y_2 - x_2y_1) + c_{34}(x_3y_4 - x_4y_3) + \dots + c_{d-1d}(x_{d-1}y_d - x_dy_{d-1}),$$

where  $c_{ij}$ s are non-zero. A simple rotation transforms  $f(x, y)$  into

$$c_{12}(x_1^2 - y_2^2 - x_2^2 + y_1^2) + \dots + c_{d-1d}(x_{d-1}^2 - y_d^2 - x_d^2 + y_{d-1}^2),$$

a non-degenerate hyperboloid, so the number of non-vanishing principal curvatures is indeed  $2d-1$ . The conclusion of Lemma 3.18 follows.

This takes care of the even dimensional case of Theorem 3.7; the odd dimensional case follows at once by combining this with Theorem 3.8.

#### 4. Applications to discrete geometry

The purpose of this section is to use a variant of a “continuous-to-discrete” mechanism, developed in [19, 21, 20, 23], to show that Thms. 3.4 and 3.7 yield interesting discrete analogues. We work in the setting of *s-adaptable* sets, defined below. These are more general than the so-called *homogeneous* sets, used widely in geometric combinatorics, e.g., [41] and the references there.

DEFINITION 4.1. ([23]) Let  $P$  be a set of  $n$  points contained in  $[0, 1]^d$ ,  $d \geq 2$ . Let  $\chi_{B^p_{n^{-\frac{1}{s}}}}(x)$  be the characteristic function of the ball of radius  $n^{-\frac{1}{s}}$  centered at  $p$ , and define the measure

$$(4.1) \quad d\mu_P^s(x) = n^{-1} \cdot n^{\frac{d}{s}} \cdot \sum_{p \in P} \chi_{B^p_{n^{-\frac{1}{s}}}}(x) dx.$$

We say that  $P$  is *s-adaptable* if

$$I_s(\mu_P^s) = \int \int |x - y|^{-s} d\mu_P^s(x) d\mu_P^s(y) < \infty.$$

This is equivalent to the statement

$$(4.2) \quad n^{-2} \sum_{p \neq p' \in P} |p - p'|^{-s} \lesssim 1.$$

To understand this condition in clearer geometric terms, suppose that  $P$  comes from a 1-separated set  $A \subset \mathbb{R}^d$ , rescaled down by its diameter so as to be contained in  $[0, 1]^d$ . Then the condition (4.2) takes the form

$$(4.3) \quad n^{-2} \sum_{a \neq a' \in A} |a - a'|^{-s} \lesssim (\text{diameter}(A))^{-s}.$$

This says  $P$  is *s-adaptable* if it is a scaled 1-separated set where the expected value of the distance between two points raised to the power  $-s$  is comparable to the value of the diameter raised to the power of  $-s$ . Thus, for a set to be *s-adaptable*, clustering is not allowed to be too severe, on average.

To put it in more technical terms, *s-adaptability* means that a discrete point set  $P$  can be thickened into a set which is uniformly *s-dimensional* in the sense that its energy integral of order  $s$  is finite. Unfortunately, it is shown in [23] that there exist finite point sets which are not *s-adaptable* for certain ranges of the parameter  $s$ . The point is that the notion of Hausdorff dimension is much more subtle than the simple “size” estimate. However, many natural classes of sets are *s-adaptable*. For example, all homogeneous sets, studied by Solymosi and Vu [41] and others, are *s-adaptable* for any  $0 < s < d$ . See also [22] where *s-adaptability* of homogeneous sets is used to extract discrete incidence theorems from Fourier type bounds.

While *s-adaptability* is a restriction, we will see below that combining this notion with analytic methods allows one to prove robust theorems, involving *neighborhoods* of a class of geometric objects, something that is not typically possible using combinatorial methods.



**4.1. Simplexes determined by discrete sets.** Before we state the discrete result that follows from Theorem 3.4, let us briefly review what is known. If  $P$  is set of  $n$  points in  $[0, 1]^2$ , let  $u_{2,2}(n)$  denote the number of times a fixed triangle can arise among points of  $P$ . It is not hard to see that

$$(4.4) \quad u_{2,2}(n) = O(n^{\frac{4}{3}}).$$

This follows from the fact that a single distance cannot arise more than  $O(n^{\frac{4}{3}})$  times, which, in turn, follows from the celebrated Szemerédi-Trotter incidence theorem; see [2] and the references there. By the pigeon-hole principle, one can conclude that

$$(4.5) \quad \#T_2(P) \gtrsim \frac{n^3}{n^{\frac{4}{3}}} = n^{\frac{5}{3}}.$$

(Recall that  $T_k(P)$  is the set of  $k$ -point configurations of  $P$ , as in Sec. 3.)

However, one can do quite a bit better as far as the lower bound on  $\#T_2(P)$  is concerned. It is shown in [2, p. 263] that

$$\#T_2(P) \gtrsim n \cdot \#\{|x - y| : x, y \in P\}.$$

Guth and Katz have recently settled the Erdős distance conjecture in a remarkable paper [17], proving that

$$\#\{|x - y| : x, y \in P\} \gtrsim \frac{n}{\log(n)};$$

from this, it follows that

$$\#T_2(P) \gtrsim \frac{n^2}{\log(n)},$$

which, up to logarithmic factors, is the optimal bound. However, the main result in [10] allowed the second and third listed authors to obtain an upper bound on  $u_{2,2}(n)$  for  $s$ -adaptable sets that is better than the one in (4.4).

Theorem 3.4 will allow us to obtain similar bounds for  $u_{k,d}(n)$ , the maximal number of times a given  $k + 1$  point configuration can arise among a set of  $n$  points in  $\mathbb{R}^d$ , for all  $1 \leq k \leq d$ , with  $d \geq 2$ . We introduce a variant of this quantity:

**DEFINITION 4.2.** Let  $P$  be a subset of  $[0, 1]^d$  consisting of  $n$  points as. For  $\delta > 0$ , define

$$u_{k,d}^\delta(n) = \#\{(x^1, x^2, \dots, x^{k+1}) \in P \times P \times \dots \times P : t_{ij} - \delta \leq |x^i - x^j| \leq t_{ij} + \delta; 1 \leq i < j \leq k + 1\},$$

where the dependence on  $t = \{t_{ij}\}$  is suppressed.

Observe that obtaining an upper bound for  $u_{k,d}^\delta(n)$  (with arbitrary  $t_{ij}$ ) immediately implies the same upper bound on  $u_{k,d}(n)$  defined above. The main result of this section is the following.

**THEOREM 4.3.** Suppose  $P \subset [0, 1]^d$  is  $s$ -adaptable for  $s = d - \frac{d-1}{2k} + a = s_{k,d} + a$  for every sufficiently small  $a > 0$ . Then for every  $b > 0$ , there exists  $C_b > 0$  such that

$$(4.6) \quad u_{k,d}^{n^{-\frac{1}{s_{k,d}}-b}}(n) \leq C_b n^{k+1-\frac{\binom{k+1}{2}}{s_{k,d}}+b}.$$

The proof follows from Theorem 3.4 in the following way. Let  $E$  denote the support of  $d\mu_P^s$ , defined as in (4.1) above. We know that if  $s > s_{k,d}$ , then

$$(4.7) \quad (\mu_P^s \times \mu_P^s \times \cdots \times \mu_P^s) \{(x^1, x^2, \dots, x^{k+1}) : t_{ij} \leq |x^i - x^j| \leq t_{ij} + \epsilon\} \lesssim \epsilon^{\binom{k+1}{2}}.$$

Taking  $\epsilon = n^{-\frac{1}{s}}$ , we see that the left hand side of (4.7) is

$$\approx n^{-(k+1)} \cdot u_{k,d}^{n^{-\frac{1}{s}}}(n)$$

and we conclude that

$$u_{k,d}^{n^{-\frac{1}{s}}}(n) \lesssim n^{k+1 - \frac{\binom{k+1}{2}}{s}},$$

which yields the desired result since  $s = s_{k,d} + a$ .

As we note above, this result is stronger than the previously known  $u_{2,2}(n) \lesssim n^{\frac{4}{3}}$ . We also see, for instance, that

$$u_{2,3}(n) \lesssim n^{\frac{9}{5}}.$$

In the range  $k \geq 2$ ,  $d \geq 3$ , to the best of our knowledge no non-trivial estimates for  $u_{k,d}(n)$  were previously known. While our results do not apply to all points sets (recall that they proved under the additional restriction that  $P$  is  $s$ -adaptable), our conclusion is stronger in that we show not just that any *single* configuration does not repeat very often, but the same holds for the  $n^{-\frac{1}{s}}$ -neighborhood of the configuration.

**4.2. Volumes determined by discrete sets.** Dumitrescu, Sharir and Toth prove the following result in [4].

**THEOREM 4.4.** *Let  $P$  be a subset of  $\mathbb{R}^d$  of cardinality  $n \gg 1$ . Let  $\text{area}(x, y, z)$  denote the area of the triangle with endpoints  $x, y, z$ . Then*

$$(4.8) \quad \#\{(x, y, z) \in P \times P \times P : \text{area}(x, y, z) = t\} \leq Cn^{\alpha_d}$$

for any  $t \neq 0$ , with

$$\alpha_2 = \frac{44}{19} \text{ and } \alpha_3 = \frac{17}{7}.$$

Application of Theorem 3.7 and method from the previous subsection yields the following result.

**THEOREM 4.5.** *Let  $P \subset [0, 1]^d$ ,  $d \geq 2$ , be an  $s$ -adaptable set for some  $s > s_d$ , where*

$$s_d = d - 1 + \frac{1}{2d} \text{ when } d \text{ is even, and } s_d = d - 1 + \frac{1}{2(d-1)} \text{ when } d \text{ is odd.}$$

*Let  $\text{vol}_d(x^1, x^2, \dots, x^{d+1})$  denote the volume of the  $d$ -dimensional simplex with the endpoints  $x^1, x^2, \dots, x^{d+1}$ . Then for  $s > s_d$ ,*

$$(4.9) \quad \#\{(x^1, \dots, x^{d+1}) : t - n^{-\frac{1}{s}} \leq \text{vol}_d(x^1, \dots, x^{d+1}) = t + n^{-\frac{1}{s}}\} \leq Cn^{d+1-\frac{1}{s}}.$$

When  $d = 2$ , the exponent on the right hand side of (4.9) equals  $3 - \frac{4}{5} = \frac{11}{5}$ , which is smaller than the exponent  $\alpha_2 = \frac{44}{19}$  in Theorem 4.4 above. Once again, we caution the reader that, although we obtain a better exponent, it is only under the hypothesis of  $s$ -adaptability.

In view of Remark 3.10 and the conversion mechanism of the section, we can prove, more generally, that if  $P \subset [0, 1]^d$ ,  $d \geq 2$ , is an  $s$ -adaptable set for some  $s > \frac{d}{2} + \frac{1}{4}$ , then

$$\#\{(x, y, z) : t - n^{-\frac{1}{s}} \leq \text{area}(x, y, z) \leq t + n^{-\frac{1}{s}}\} \lesssim n^{3-\frac{1}{s}}.$$

As seen above, in two dimensions this gives a slightly better exponent, in the context of  $s$ -adaptable sets, than the one obtained in [4], while in three dimensions, our exponent and the one in [4] match. In higher dimensions, the results here are the only ones currently known.

**4.3. Angles determined by discrete sets.** The results of this subsection are contained in [25], but are included to indicate the wide applicability of our method.

The following results were obtained by Pach and Sharir [32], and Apfelbaum and Sharir [1]. In [32], it is shown that for a set of  $n$  points in  $\mathbb{R}^2$ , no angle can occur more than  $cn^2 \log n$  times. Since there are about  $n^3$  triples of points, this implies that there must be at least  $c \frac{n}{\log n}$  distinct angles. In [1], it is shown that for a set of  $n$  points in  $\mathbb{R}^3$ , no angle can occur more than  $cn^{\frac{7}{3}}$  times, which gives a lower bound of at least  $cn^{\frac{2}{3}}$  distinct angles. They also show that for a set of  $n$  points in  $\mathbb{R}^4$ , no angle besides  $\frac{\pi}{2}$  can occur more than  $cn^{\frac{5}{2}}\beta(n)$  times, where  $\beta(n)$  grows extremely slowly with respect to  $n$ . This means that there must be about  $n^{\frac{1}{2}}(\beta(n))^{-1}$  distinct angles.

In dimensions four and higher, no results are currently available. We have the following theorem, which follows from Theorem 3.12 and the conversion mechanism of this section.

**THEOREM 4.6.** *Let  $P \subset \mathbb{R}^d$ ,  $\#P = n$ ,  $d \geq 2$ , be an  $s$ -adaptable set for  $s > \frac{d+1}{2}$ . Then*

$$\#\{(x^1, x^2, x^3) \in P \times P \times P : \theta_0 - n^{-\frac{1}{s}} \leq \theta(x^1, x^2, x^3) \leq \theta_0 + n^{-\frac{1}{s}}\} \lesssim n^{3-\frac{1}{s}}.$$

In dimensions two and three, these exponents are not as good as the results of [1, 32]. However, Theorem 4.6 gives non-trivial exponents in all dimensions.

We have considered three problems in this section: distribution of simplexes, distribution of volumes and distribution of angles. Many other geometric problems can be handled by similar methods. Moreover, combining classical combinatorial techniques with the methods of this section should lead to sharper exponents in many cases. We hope to address these issues in a sequel.

## 5. Nontranslation invariant multilinear estimates

**THEOREM 5.1.** *Define a variable coefficient multilinear operator by*

$$S_\mu(f_1, \dots, f_k)(x) = \int \dots \int f_1(u^1) \dots f_k(u^k) d\mu_x(u^1, \dots, u^k),$$

where  $x, u^j \in \mathbb{R}^d$  and  $\{\mu_x\}$  are non-negative Borel measures such that the map  $x \mapsto \mu_x$  is measurable. Let  $\psi$  be a smooth function supported in the double of the unit ball that is equal to one on the unit ball. Suppose that for some  $\gamma_1, \gamma_2 > 0$  we have

$$(5.1) \quad \sup_{R>0} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \hat{\mu}_x(\xi^1, \xi^2, 0 \dots, 0) \psi(x/R) dx \right| d\xi^2 \lesssim (1 + |\xi^1|)^{-\gamma_1}$$

$$(5.2) \quad \sup_{R>0} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \hat{\mu}_x(\xi^1, \xi^2, 0 \dots, 0) \psi(x/R) dx \right| d\xi^1 \lesssim (1 + |\xi^2|)^{-\gamma_2}$$

Then, acting on nonnegative functions,  $S_\mu$  is a bounded multilinear form,

$$S_\mu : L^2_{-\frac{\gamma_1}{2}}(\mathbb{R}^d) \times L^2_{-\frac{\gamma_2}{2}}(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \times \cdots \times L^\infty(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d).$$

Moreover, the same conclusion follows when (5.1) and (5.2) are replaced by the more symmetric condition

$$(5.3) \quad \sup_{R>0} \left[ \sup_{\xi^1 \in \mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |\xi^1|)^{\frac{\gamma_1}{2}} (1 + |\xi^2|)^{\frac{\gamma_2}{2}} |Q_R(\xi^1, \xi^2)| d\xi^2 + \sup_{\xi^2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |\xi^1|)^{\frac{\gamma_1}{2}} (1 + |\xi^2|)^{\frac{\gamma_2}{2}} |Q_R(\xi^1, \xi^2)| d\xi^1 \right] < \infty,$$

where

$$Q_R(\xi^1, \xi^2) = \int_{\mathbb{R}^d} \hat{\mu}_x(\xi^1, \xi^2, 0, \dots, 0) \psi(x/R) dx.$$

REMARK 5.2. In Theorem 5.1 (and, as noted previously, in Theorem 1.2) there is nothing special about the first two coordinates in the assumption of Fourier decay of the measure. The theorems can be stated in terms of any two distinct, distinguished coordinates, and correspondingly change the resulting boundedness conclusion, as in the following.

**5.1. Proof of Theorem 5.1.** We may assume that  $f_j$  are nonnegative Schwartz functions. Then since  $\psi$  is equal to 1 on the unit ball, for any  $R$  we have

$$\begin{aligned} \|H_\mu(f_1, \dots, f_k)\|_{L^1(B_R)} &\leq \prod_{j=3}^k \|f_j\|_\infty \cdot \int \cdots \int f_1(u^1) f_2(u^2) d\mu_x(u^1, \dots, u^k) \psi(x/R) dx \\ &= \prod_{j=3}^k \|f_j\|_\infty \int \int \int \hat{f}_1(\xi^1) \hat{f}_2(\xi^2) \hat{\mu}_x(\xi^1, \xi^2, 0, \dots, 0) \psi(x/R) dx d\xi^1 d\xi^2 \\ &\leq \prod_{j=3}^k \|f_j\|_\infty \cdot I \cdot II, \end{aligned}$$

where

$$I^2 = \int \int |\hat{f}_1(\xi^1)|^2 \left| \int \hat{\mu}_x(\xi^1, \xi^2, 0, \dots, 0) \psi(x/R) dx \right| d\xi^2 d\xi^1$$

and

$$II^2 = \int \int |\hat{f}_2(\xi^2)|^2 \left| \int \hat{\mu}_x(\xi^1, \xi^2, 0, \dots, 0) \psi(x/R) dx \right| d\xi^1 d\xi^2.$$

By assumption (5.1),

$$I^2 \lesssim \int |\hat{f}_1(\xi^1)|^2 (1 + |\xi^1|)^{-\gamma_1} d\xi^1,$$

and by assumption (5.2),

$$II^2 \lesssim \int |\hat{f}_2(\xi^2)|^2 (1 + |\xi^2|)^{-\gamma_2} d\xi^2.$$

To obtain the same conclusion of the theorem under assumption (5.3), we slightly modify the preceding proof. We multiply and divide the integrand in the expression

$$\prod_{j=3}^k \|f_j\|_\infty \int \int \hat{f}_1(\xi^1) \hat{f}_2(\xi^2) Q_R(\xi^1, \xi^2) d\xi^1 d\xi^2$$

by  $(1 + |\xi^1|)^{\frac{\gamma_1}{2}}(1 + |\xi^2|)^{\frac{\gamma_2}{2}}$ . Then we apply the Cauchy-Schwarz inequality with respect to the measure

$$(1 + |\xi^1|)^{\frac{\gamma_1}{2}}(1 + |\xi^2|)^{\frac{\gamma_2}{2}} |Q_R(\xi^1, \xi^2)| d\xi^1 d\xi^2$$

and we use condition (5.3) to conclude the proof.

**COROLLARY 5.3.** *Under the assumptions of Theorem 5.1, we have that*

$$S_\mu : L^{p_{\gamma_1}}(\mathbb{R}^d) \times L^{p_{\gamma_2}}(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \times \cdots \times L^\infty(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d),$$

for  $\gamma_j < d$ ,  $j = 1, 2$ , and

$$(5.4) \quad p_{\gamma_j} = \frac{2}{1 + \frac{\gamma_j}{d}} > 1.$$

The corollary follows from Theorem 5.1 by applying the Hardy-Littlewood-Sobolev embedding of  $L^q$  into  $L^p_{-s}$ , where  $s > 0$  and  $\frac{1}{q} = \frac{1}{p} + \frac{s}{d}$ ,  $1 < q < p < \infty$ , together with the observation that, if the  $L^p$  improving property holds for nonnegative functions, then it holds for all functions.

**COROLLARY 5.4.** *Suppose that for  $j, \ell \in \{1, \dots, k\}$ ,  $\Xi_{j,\ell}$  is an ordered  $k$ -tuple with  $\xi^j$  in the  $j$ th entry,  $\xi^\ell$  in the  $\ell$ th entry and 0 in the remaining entries. Let  $0 < \gamma_j < d$  for  $j = 1, \dots, k$  and*

$$Q_R^{j,\ell}(\Xi_{j,\ell}) = \int_{\mathbb{R}^d} \hat{\mu}(\Xi_{j,\ell}) \psi(x/R) dx.$$

Assume that either for all  $j, \ell \in \{1, \dots, k\}$  we have

$$(5.5) \quad \sup_{R>0} \int_{\mathbb{R}^d} |Q_R^{j,\ell}(\Xi_{j,\ell})| d\xi^\ell \lesssim (1 + |\xi^j|)^{-\gamma_j},$$

or for all  $j, \ell \in \{1, \dots, k\}$ ,  $j \neq \ell$ , we have

$$(5.6) \quad \sup_{R>0} \left[ \sup_{\xi^j \in \mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |\xi^j|)^{\frac{\gamma_j}{2}} (1 + |\xi^\ell|)^{\frac{\gamma_\ell}{2}} |Q_R(\Xi_{j,\ell})| d\xi^\ell + \sup_{\xi^\ell \in \mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |\xi^j|)^{\frac{\gamma_j}{2}} (1 + |\xi^\ell|)^{\frac{\gamma_\ell}{2}} |Q_R(\Xi_{j,\ell})| d\xi^j \right] < \infty.$$

Then the following Lebesgue-space estimate holds for  $S_\mu$ :

$$S_\mu : L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_k}(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d),$$

for any  $1 < p_j < 2$ ,  $1 \leq j \leq k$ , and  $\theta_j$  with  $0 \leq \theta_j \leq 1$ ,  $\sum_{j=1}^k \theta_j = 1$ , such that

$$(5.7) \quad \sum_{j=1}^k \frac{1}{p_j} = 1 + \frac{1}{d} \sum_{j=1}^k \theta_j \gamma_j.$$

This follows from Cor. 5.3 by permuting the placement of the Sobolev spaces in all possible pairs of locations and applying multilinear complex interpolation between Lebesgue spaces. More precisely, let  $\vec{E}_l$  be the vector having 1 in the  $l$ th entry and zero elsewhere. The initial points of the interpolation are  $\frac{1}{p_{\gamma_j}} \vec{E}_j + \frac{1}{p_{\gamma_l}} \vec{E}_l$ ,  $1 \leq j \neq l \leq k$ , and the intermediate point is

$$(5.8) \quad \left( \frac{1}{p_1}, \dots, \frac{1}{p_k} \right) = \sum_{\substack{1 \leq j, l \leq k \\ j \neq l}} \theta_{j,l} \left( \frac{1}{p_{\gamma_j}} \vec{E}_j + \frac{1}{p_{\gamma_l}} \vec{E}_l \right)$$

for some  $0 \leq \theta_{j,l} = \theta_{l,j} \leq 1$  with  $\sum_{j \neq l} \theta_{j,l} = 1$ . Relationship (5.7) follows from (5.8) using (5.4) setting  $\theta_j = \frac{1}{2} \sum_{s \neq j} \theta_{j,s}$  and noting that  $\frac{1}{p_j} = \sum_{s \neq j} \theta_{j,s} \frac{1}{p_{\gamma_j}}$ .

**5.2. Sharpness of estimates.** We now show that the bound given by Cor. 5.3 is, in general, sharp. To see this consider the bilinear fractional integration operator

$$B_\gamma(f, g)(x) = \int \int f(x-u)g(x-v)(|u|^2 + |v|^2)^{-\frac{2d-\gamma}{2}} dudv.$$

(See [12, 27, 13] for more singular operators of fractional integral type.) It is not difficult to check that  $B_\gamma$  satisfies the assumption of Theorem 5.1. Replacing  $f(x), g(x)$  by  $f(\delta x), g(\delta x)$  and changing variables shows that if

$$B_\gamma : L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \rightarrow L^r(\mathbb{R}^d),$$

then

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{r} \leq \frac{\gamma}{d}.$$

Plugging in  $p = q$  and  $r = 1$  shows that the conclusion of Cor. 5.3 is, in general sharp.

One easily checks that the operator  $A_2^2$  from (1.7) satisfies the assumption of Theorem 5.1 with  $\gamma = \frac{1}{2}$ . This yields  $p_{\gamma_j} = \frac{8}{5}$ ,  $j = 1, 2$  in (5.4). However, it is not difficult to check that the better bound  $L^{\frac{3}{2}}(\mathbb{R}^2) \times L^{\frac{3}{2}}(\mathbb{R}^2) \rightarrow L^1(\mathbb{R}^2)$  actually holds.

This is in contrast to the situation in the linear case. The direct analog of a  $L^p \times L^p \rightarrow L^1$  bound in the bilinear case is a  $L^p \rightarrow L^2$  bound in the linear case. Let  $A_1^d f(x)$  be defined as in (1.2) above. Since  $|\widehat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$  by the method of stationary phase, it follows that

$$\begin{aligned} \|A_1^d f\|_2 &= \|\widehat{A_1^d f}\|_2 \lesssim \left( \int |\widehat{f}(\xi)|^2 (1 + |\xi|)^{-(d-1)} d\xi \right)^{\frac{1}{2}} \\ &\lesssim \|f\|_{L^{\frac{2}{2-d}}(\mathbb{R}^d)} \end{aligned}$$

by the classical Hardy-Littlewood-Sobolev inequality (see, e.g., [39]). This is precisely the sharp  $L^p(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  bound for the spherical averaging operator  $A_1^d$ , as pointed out following (1.4).

## 6. Estimates for multilinear adjoints of translation invariant multilinear generalized Radon transforms

We will now focus our attention on the translation invariant case where we can write the multilinear generalized Radon transforms as

$$T_\mu(f_1, \dots, f_k)(x) = \int \dots \int f_1(x-u^1) \dots f_k(x-u^k) d\mu(u^1, \dots, u^k)$$

where  $\mu$  is a nonnegative Borel measure. Define the  $i$ -th multilinear adjoint  $T_\mu^{*i}$  by

$$\langle T_\mu(f_1, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_k), f_{k+1} \rangle = \langle T_\mu^{*i}(f_1, \dots, f_{i-1}, f_{k+1}, f_{i+1}, \dots, f_k), f_i \rangle$$

where  $\langle \cdot, \cdot \rangle$  is the  $L^2(\mathbb{R}^d)$  inner product.

**THEOREM 6.1.**

(1) *Suppose that*

$$|\widehat{\mu}(-\xi, \xi, 0, \dots, 0)| \lesssim (1 + |\xi|)^{-\gamma}$$

*for some  $\gamma > 0$ . Then for all  $\gamma_1, \gamma_2 > 0$  such that  $\gamma = \gamma_1 + \gamma_2$  we obtain the following estimate on nonnegative functions*

$$T_\mu^{*i} : L^2_{-\frac{\gamma_1}{2}}(\mathbb{R}^d) \times L^2_{-\frac{\gamma_2}{2}}(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \times \dots \times L^\infty(\mathbb{R}^d)$$

for  $i = 3, 4, \dots, k$ .

(2) Suppose that

$$(6.1) \quad |\widehat{\mu}(0, \xi, 0, \dots, 0)| \lesssim (1 + |\xi|)^{-\gamma}$$

for some  $\gamma > 0$ . Then for all  $\gamma_1, \gamma_2 > 0$  such that  $\gamma = \gamma_1 + \gamma_2$  we obtain the following estimate on nonnegative functions

$$T_{\mu}^{*1} : L_{-\frac{\gamma_1}{2}}^2(\mathbb{R}^d) \times L_{-\frac{\gamma_2}{2}}^2(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \times \dots \times L^\infty(\mathbb{R}^d)$$

(3) Suppose that

$$(6.2) \quad |\widehat{\mu}(\xi, 0, 0, \dots, 0)| \lesssim (1 + |\xi|)^{-\gamma}$$

for some  $\gamma > 0$ . Then for all  $\gamma_1, \gamma_2 > 0$  such that  $\gamma = \gamma_1 + \gamma_2$  we obtain the following estimate on nonnegative functions

$$T_{\mu}^{*2} : L_{-\frac{\gamma_1}{2}}^2(\mathbb{R}^d) \times L_{-\frac{\gamma_2}{2}}^2(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \times \dots \times L^\infty(\mathbb{R}^d)$$

To prove Theorem 6.1, start by noting that it is easy to see that  $T_{\mu}^{*i}(f_1, \dots, f_k)(x)$  is equal to

$$\int \dots \int f_1(x+u^i-u^1) \dots f_{i-1}(x+u^i-u^{i-1}) f_i(x+u^i) f_{i+1}(x+u^i-u^{i+1}) \dots f_k(x+u^i-u^k) d\mu(u^1, \dots, u^k)$$

for all  $i = 1, \dots, k$ . We now proceed to prove all the cases in the theorem. Assume  $f_j$  are nonnegative Schwartz functions.

When  $i > 2$  we get

$$\begin{aligned} \|T_{\mu}^{*i}(f_1, \dots, f_k)\|_{L^1(\mathbb{R}^d)} &\leq \prod_{j=3}^k \|f_j\|_{\infty} \cdot \int \dots \int f_1(x+u^i-u^1) f_2(x+u^i-u^2) dx d\mu(u^1, \dots, u^k) \\ &= \prod_{j=3}^k \|f_j\|_{\infty} \cdot \int \dots \int f_1(y) f_2(x+u^1-u^2) dy d\mu(u^1, \dots, u^k) \end{aligned}$$

and we observe that precisely this quantity came up in the proof of Theorem 1.2. Since we have the same assumptions as in that theorem then we note that the same proof will work.

When  $i = 1$  we get

$$\begin{aligned} \|T_{\mu}^{*i}(f_1, \dots, f_k)\|_{L^1(\mathbb{R}^d)} &\leq \prod_{j=3}^k \|f_j\|_{\infty} \cdot \int \dots \int f_1(x+u^1) f_2(x+u^1-u^2) dx d\mu(u^1, \dots, u^k) \\ &= \prod_{j=3}^k \|f_j\|_{\infty} \cdot \int \dots \int f_1(y) f_2(y-u^2) dy d\mu(u^1, \dots, u^k) \\ &= \prod_{j=3}^k \|f_j\|_{\infty} \cdot \int \dots \int f_1(y) \int \widehat{f_2}(\xi) e^{2\pi i \xi \cdot (y-u^2)} d\xi dy d\mu(u^1, \dots, u^k) \\ &= \prod_{j=3}^k \|f_j\|_{\infty} \int \widehat{f_1}(-\xi) \widehat{f_2}(\xi) \widehat{\mu}(0, \xi, 0, \dots, 0) d\xi \end{aligned}$$

$$\begin{aligned}
&\leq \prod_{j=3}^k \|f_j\|_\infty \int \left| \widehat{f_1}(-\xi) \right| \left| \widehat{f_2}(\xi) \right| (1 + |\xi|)^\gamma d\xi \\
&\leq \prod_{j=3}^k \|f_j\|_\infty \left( \int \left| \widehat{f_1}(\xi) \right|^2 (1 + |\xi|)^{-\gamma_1} d\xi \right) \left( \int \left| \widehat{f_2}(\xi) \right|^2 (1 + |\xi|)^{-\gamma_2} d\xi \right)
\end{aligned}$$

where in the second to last step we used assumption (6.1). The case  $i = 2$  is similar.

## 7. Regular value theorem in a fractal setting

The regular value theorem in elementary differential geometry says that if  $\phi : X \rightarrow Y$ , where  $X$  is a smooth manifold of dimension  $n$  and  $Y$  is a smooth manifold of dimension  $m < n$ , then if  $\phi$  is a submersion on the set

$$\{x \in X : \vec{\phi}(x) = y\},$$

where  $y$  is a fixed element of  $Y$ , then the set

$$\vec{\phi}^{-1}(y) = \{x \in X : \vec{\phi}(x) = y\}$$

is either empty or is a  $(n - m)$ -dimensional sub-manifold of  $X$ .

In [7], the authors considered the situation where  $Y = \mathbb{R}^m$  and  $X$  is replaced by  $E \times E$ , where  $E \subset \mathbb{R}^d$  is a set of a given Hausdorff dimension, which, in general, is far from being a smooth manifold. A direct analogue of the regular value theorem would be the statement that the set

$$S_t^{\vec{\phi}}(E_1, E_2) = \{(x, y) \in E_1 \times E_2 : \phi_l(x, y) = t_l; 1 \leq l \leq m\}$$

is either empty or has fractal dimension exactly  $s_1 + s_2 - m$ , where  $s_j$  is the Hausdorff dimension of  $E_j$ . The examples in [7], based on arithmetic constructions, show that the lower bound does not in general hold due to the fractal nature of the problem, it is shown in [7] that if  $m = 1$  and the Monge-Ampère determinant

$$(7.1) \quad \det \begin{pmatrix} 0 & \nabla_x \phi \\ -(\nabla_y \phi)^T & \frac{\partial^2 \phi}{\partial x_i \partial y_j} \end{pmatrix} \neq 0$$

on the set  $\{(x, y) : \phi(x, y) = t\}$ , then the upper Minkowski dimension of  $S_t^{\vec{\phi}}(E_1, E_2)$  is indeed  $\leq s_1 + s_2 - 1$ .

The multilinear machinery developed in this paper allows us to study the upper Minkowski dimension of the set

$$S_{t,k}^{\Phi}(E_1, \dots, E_{k+1}) = \{(x^1, \dots, x^{k+1}) \in E_1 \times E_2 \times \dots \times E_{k+1} : \Phi(x^1, \dots, x^{k+1}) = t\},$$

where  $\Phi$  is defined in (3.1) above.

The techniques in [7] show readily that Theorem 3.4 implies the following result.

**THEOREM 7.1.** *Let  $E_j \subset [0, 1]^d$  of Hausdorff dimension  $s_j$ . Under the assumptions of Theorem 3.2,*

$$(7.2) \quad \overline{\dim}_{\mathcal{M}}(S_{t,k}^{\Phi}(E_1, \dots, E_{k+1})) \leq s_1 + s_2 + \dots + s_{k+1} - n,$$

*provided that*

$$s_1 + s_2 + \dots + s_{k+1} > (k + 1) \left( d - \frac{\gamma}{k} \right).$$



REMARK 7.2. The critical exponent provided by Theorem 7.1 is known to be sharp in the case  $k = 1$ . In the multilinear case, the issue is sharpness is related to some interesting questions about the distribution of lattice points on varieties of higher co-dimension. This question shall be investigated systematically in a sequel.

REMARK 7.3. One can use Theorems 3.4 and 3.7 and 3.12 to provide corollaries of Theorem 7.1 in a variety of settings.

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